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The influence of jet flow on jet noise. Part 2. The noise of heated jets

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This paper continues the study of part 1 into the area of the noise of heated jets. First, this part of the study discusses how a convected wave equation approach based on Lilley's equation leads to additional dipole and simple source terms associated with the velocity fluctuations due to transverse gradients of the mean density of the flow. Once these source terms have been identified and roughly estimated, we revert to a plug-flow model of the jet flow (where now the jet temperature and jet density differ from the ambient values) to estimate the radiation of these singularities. Several novel physical aspects of hot-jet noise are uncovered by the analysis. Indeed the problem of hot-jet noise is the one where the greatest deviations from Lighthill's ideas on jet noise generation are evident. The results are applied to available data and a very satisfactory measure of agreement is obtained with respect to the various predictions of the theory. Mechanisms for 'excess' pure jet noise scaling on M^6 and M^4 are found to result from the density gradients of the mean flow. The satisfactory agreement with the data suggests a solution of the problem of scaling jet noise with regard to jet temperature effects. The ability to predict correctly the data also suggests that the jet temperature has very little effect on the turbulence source spectrum generating jet noise at least for jet exit velocities up to about 1·5 times the atmospheric speed of sound.

1. Interpretation of Lilley's equation for heated jets

Lilley's equation can be written in terms of the fluctuating pressure p' as

$$\frac{1}{\overline{\overline{a^2}}}\frac{\bar{D}^3p'}{\bar{D}t^3}-\frac{\bar{D}}{\bar{D}t}(\nabla^2p')-\frac{d}{dr}\log\overline{(a^2)}\frac{\bar{D}}{\bar{D}t}\left(\frac{\partial p'}{\partial r}\right)+2\frac{dV_1}{dr}\frac{\partial^2p'}{\partial x_1\,\partial r}=\overline{\rho(r)}\left\{\frac{\bar{D}}{\bar{D}t}\frac{\partial^2}{\partial x_i\,\partial x_j}(u_i'u_j')\right\}, \tag{1}$$

where for simplicity the shear-noise term of Lilley's original equation has been dropped. Consider the source term $\overline{\rho(r)}\,\partial^2Q_{ij}/\partial x_i\,\partial x_j$, where $Q_{ij}=u_i'u_j'$. As before, let $Q_{ij}=Q_{ij}\,\delta(x-Vt)\,\delta(y-y_0)\,\delta(z-z_0)$. The assertion now is that a quadrupole source term of the form

$$\overline{\rho(r)}\,\partial^2[\delta(y-y_0)\,\delta(z-z_0)\,\delta(x-Vt)]/\partial x_i\,\partial x_j$$

for either (or both of) i or $j=2$ or 3 contains additional source-like dipole-like terms. Consider for example the term

$$\overline{\rho(r)}\,\partial^2[\delta(y-y_0)\,\delta(z-z_0)\,\delta(x-Vt)]/\partial y\,\partial x. \tag{2}$$

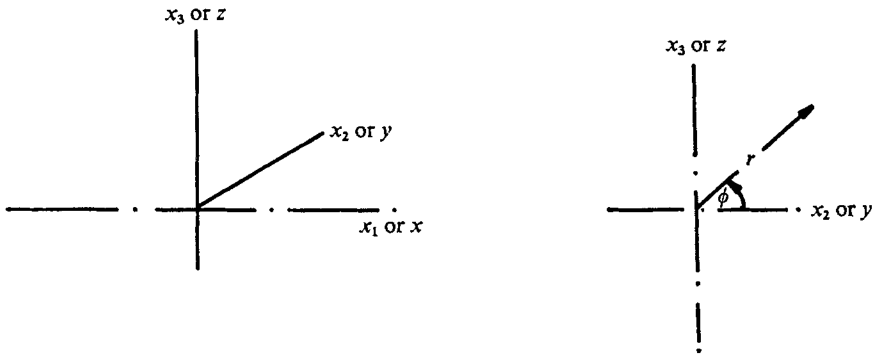

FIGURE 1

We first remind the reader of some results for generalized functions (see, for example, Lighthill 1959):

$$f(x)\,\delta(x) = f(0)\,\delta(x), \tag{3a}$$

$$f(x)\,\delta'(x) = f(0)\,\delta'(x) - f'(0)\,\delta(x), \tag{3b}$$

$$f(x)\,\delta''(x) = f(0)\,\delta''(x) - 2f'(0)\,\delta'(x) + f''(0)\,\delta(x). \tag{3c}$$

Using (3b) we may show that

$$\overline{\rho(r)}\frac{\partial^2}{\partial y\,\partial x}[\delta(x-Vt)\,\delta(z-z_0)\,\delta(y-y_0)] = \overline{\rho(r_0)}\frac{\partial^2}{\partial y\,\partial x}[\delta(x-Vt)\,\delta(z-z_0)\,\delta(y-y_0)] - \overline{\frac{\partial\rho}{\partial y}(r=r_0)}\frac{\partial}{\partial x}[\delta(x-Vt)\,\delta(z-z_0)\,\delta(y-y_0)]. \tag{4}$$

This shows that the quadrupole term associated with (2) is in fact a combination of a traditional x-y quadrupole term proportional to the mean density at $y = y_0$, $z = z_0$ and an axial dipole term proportional to the mean density gradient at $y = y_0$, $z = z_0$. Note that (according to the definition sketch in figure 1)

$$\overline{\frac{\partial\rho}{\partial y}} = \cos\phi\,\overline{\frac{d\rho}{dr}(r=r_0)}.$$

The result (4) is of such importance to the problem of heated-jet noise that it is worth explaining it in at least two other, alternative ways. First, consider any differential equation of the form

$$Lp' = h(y,z)\,\partial^2 f/\partial y\,\partial z. \tag{5}$$

To solve this equation (where, for convenience, L is assumed to be a linear operator involving only y and z), one could construct the Green's function $g(y-y_0, z-z_0)$ satisfying

$$\mathscr{L}g = \delta(z-z_0)\,\delta(y-y_0), \tag{6}$$

so that

$$p' = \iint g h(y_0, z_0)\frac{\partial^2 f}{\partial y_0 \partial z_0}\,dy_0\,dz_0. \tag{7}$$

Integrating by parts and with suitable restrictions on g and its derivatives at y_0 and $z_0 = \pm\infty$,

$$p' = \iint f\frac{\partial^2}{\partial y_0 \partial z_0}(gh)\,dy_0\,dz_0 \tag{8}$$

$$= \iint f\left[h\frac{\partial^2 g}{\partial y_0\,\partial z_0} + \frac{\partial h}{\partial z_0}\frac{\partial g}{\partial y_0} + \left(\frac{\partial h}{\partial y_0}\right)\left(\frac{\partial g}{\partial z_0}\right) + g\frac{\partial^2 h}{\partial y_0\,\partial z_0}\right]dy_0\,dz_0. \tag{9}$$

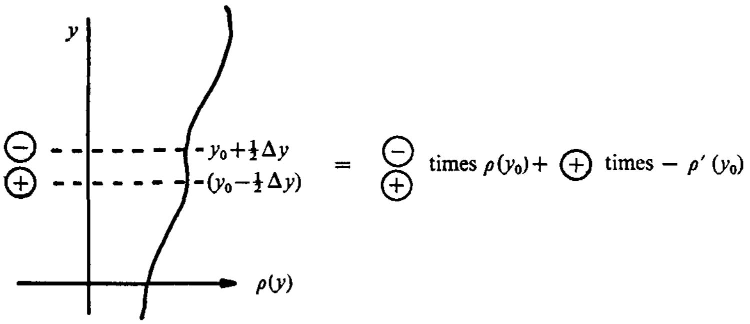

FIGURE 2

Such a procedure is, in fact, the basis for deriving results such as (3) or (4). Expression (9) clearly shows the generation of lower-order singular solutions (proportional to $\partial g/\partial y_0$, $\partial g/\partial z_0$ and g) in addition to one proportional to $\partial^2 g/\partial z_0 \partial y_0$ owing to the gradients $\partial h/\partial z_0$, $\partial h/\partial y_0$ and $\partial^2 h/\partial y_0 \partial z_0$.

Looking at it in another way, whether one considers Lilley's equation or Phillips' equation

$$\frac{\bar{D}^2 r'}{\gamma p_A \bar{D}t^2} - \frac{\partial}{\partial x_i}\left(\frac{1}{\bar{\rho}}\frac{\partial r'}{\partial x_i}\right) = \frac{1}{p_A}\left\{\frac{2\partial \bar{u}_k}{\partial x_j}\frac{\partial u_j'}{\partial x_k} + \frac{\partial}{\partial x_i} v_j'\left(\frac{\partial v_i'}{\partial x_j}\right)\right\}, \tag{10}$$

the fundamental solution to the above (i.e. with a $\delta(y-y_0)\,\delta(z-z_0)$ type of term on the right-hand side) is proportional to $\bar{\rho}(r_0)$ owing to the appearance of the term $\bar{\rho}^{-1}$, associated with the highest transverse-derivative term $\nabla^2 r'$ on the left-hand side of (10). This characteristic leads to the generation of lower-order singularities when density gradients are considered.

Figure 2 illustrates how a transverse dipole term proportional to $\rho(y)$ will produce both a dipole term proportional to $\rho(y_0)$ and a simple source term proportional to $-\rho'(y_0)$.

Enough has now been said concerning the role of mean density gradients in Lilley's equation to make the following observations (using Lilley's equation in the form (1)).

(*a*) The purely axial x-x quadrupole generates no lower-order singularities (since the mean density gradients are assumed to be purely transverse).

(*b*) The x-y and x-z quadrupoles resolve into a purely quadrupole term proportional to the local jet density and an axial-dipole term proportional to the local density gradient. This is summed up by

$$\begin{aligned}\overline{\rho(r)}\frac{\partial^2}{\partial x\,\partial y}[\delta(y-y_0)\,\delta(z-z_0)\,\delta(x-Vt)] &= \overline{\rho(r_0)}\frac{\partial^2}{\partial x\,\partial y}[\delta(y-y_0)\,\delta(z-z_0)\,\delta(x-Vt)] \\ &\quad -\cos\phi\,\overline{\left(\frac{d\rho}{dr}\right)(r=r_0)}\frac{\partial}{\partial x}[\delta(y-y_0)\,\delta(z-z_0)\,\delta(x-Vt)] \end{aligned}\tag{11}$$

and similarly for the x-z quadrupole.

(*c*) The y-y, y-z and z-z quadrupoles generate both transverse dipoles (proportional to $\overline{d\rho/dr}$) and simple sources (proportional to $\overline{d^2\rho/dr^2}$ and $r^{-1}\overline{d\rho/dr}$) in

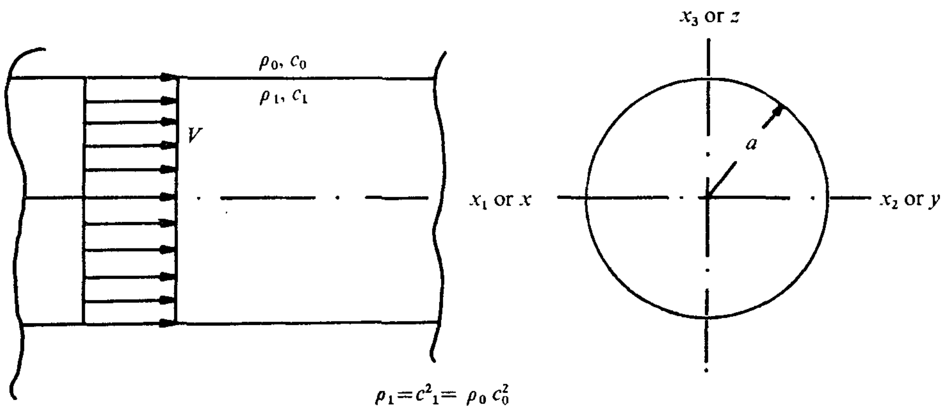

FIGURE 3

addition to quadrupoles (proportional to $\overline{\rho(r)}$). These aspects can be summed up by

$$\begin{aligned}\overline{\rho(r)}\frac{\partial^2}{\partial y^2}\delta(x-Vt)\,\delta(z-z_0)\,\delta(y-y_0) &= \overline{\rho(r_0)}\frac{\partial^2}{\partial y^2}[\delta(x-Vt)\,\delta(z-z_0)\,\delta(y-y_0)]\\ &-2\cos\phi\,\overline{\left(\frac{d\rho}{dr}\right)(r=r_0)}\frac{\partial}{\partial y}[\delta(x-Vt)\,\delta(z-z_0)\,\delta(y-y_0)]\\ &+\left(\cos^2\phi\,\overline{\frac{d^2\rho}{dr^2}(r=r_0)}+\frac{\sin^2\phi}{r_0}\overline{\frac{d\rho}{dr}}(r=r_0)\right)\delta(x-Vt)\,\delta(z-z_0)\,\delta(y-y_0),\end{aligned}\tag{12}$$

an analogous result for the z-z quadrupole and

$$\begin{aligned}\overline{\rho(r)}\frac{\partial^2}{\partial y\,\partial z}[\delta(x-Vt)\,\delta(z-z_0)\,\delta(y-y_0)] &= \overline{\rho(r_0)}\frac{\partial^2}{\partial y\,\partial z}[\delta(x-Vt)\,\delta(z-z_0)\,\delta(y-y_0)]\\ &-\overline{\frac{d\rho}{dr}(r=r_0)}\left\{\sin\phi\frac{\partial}{\partial y}[\delta(x-Vt)\,\delta(z-z_0)\,\delta(y-y_0)]\right.\\ &\left.+\cos\phi\frac{\partial}{\partial z}[\delta(x-Vt)\,\delta(z-z_0)\,\delta(y-y_0)]\right\}\\ &+\frac{\sin(2\phi)}{2}\left\{\overline{\frac{d^2\rho}{dr^2}(r=r_0)}-\frac{1}{r_0}\overline{\frac{d\rho}{dr}(r=r_0)}\right\}\delta(x-Vt)\,\delta(z-z_0)\,\delta(y-y_0).\end{aligned}\tag{13}$$

Several of these aspects of the role of mean density gradients have also been pointed out by Morfey (1973) though not from the point of view of Lilley's equation but from the point of view of consequences of using relation (6) of part 1 rather than a relation of the form $p' = c^2\rho'$.

2. Method of solution

Once the additional singularities generated by the presence of mean density gradients have been identified we may use (1) with a plug-flow model of the jet flow as shown in figure 3. We note that $\rho_1 c_1^2$ must be equal to $\rho_0 c_0^2$ owing to the constancy of the mean static pressure. Equation (1) appears to be a preferred form of Lilley's equation in that in (1) the coefficient of the highest transverse-derivative term involving p', namely $\nabla^2 r'$, is unity. As in part 1 we shall deal with centre-line eddy convection and assume V_1 to be equal to V_c. One important qualifier, however, is that it would not be meaningful to estimate $\overline{\rho(r)}$, $\overline{d\rho/dr}$, $\overline{d^2\rho/dr^2}$ and $r^{-1}\overline{d\rho/dr}$ by their values at the jet centre-line ($r_0 = 0$). To be consistent with a plug-flow model, we must use some average representative estimates of $\bar{\rho}$, $\overline{d\rho/dr}$, etc. Some physical judgement is involved in this process and we shall discuss this matter in the next section.

The quadrupole singular solutions themselves can be derived in a manner very similar to that for those derived in part 1; the procedure is briefly illustrated for the x-x quadrupole. With a plug-flow model, we have to solve

$$\frac{1}{c_1^2}\frac{\bar{D}^2 p'}{\bar{D}t^2} - \nabla^2 p' = \rho_1 Q_{xx}^0 \frac{\partial^2}{\partial x^2}\exp(i\omega_0 t)\,[\delta(x-Vt)\,\delta(y)\,\delta(z)] \quad \text{for} \quad 0 \leqslant r < a \tag{14}$$

and

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0 \quad \text{for} \quad r > a \tag{15}$$

with both p' and η continuous at $r = a$,

$$\frac{\bar{D}^2\eta}{\bar{D}t^2} = \frac{-1}{\rho_1}\frac{\partial p'}{\partial r} \quad \text{for} \quad 0 \leqslant r < a \tag{16}$$

and

$$\frac{\partial^2\eta}{\partial t^2} = \frac{-1}{\rho_0}\frac{\partial p'}{\partial r} \quad \text{for} \quad r > a. \tag{17}$$

The problem for the axial Fourier transforms P' and N then is

$$\nabla_{y,z}^2 P' + (k_1^2 - \alpha^2)\,P' = \rho_1 Q_{xx}^0 \alpha^2 \delta(y)\,\delta(z), \tag{18}$$

where $k_1 = \omega_0/c_1$ for $0 \leqslant r < a$, and

$$\nabla_{y,z}^2 P' + [(k_0 + \alpha M_0)^2 - \alpha^2]\,P' = 0 \quad \text{for} \quad r > a, \tag{19}$$

where $k_0 = \omega_0/c_0$ and $M_0 = V/c_0$. At $r = a$, P' should be continuous as should N (the transform of η'), where

$$N = \frac{1}{\rho_1\omega_0^2}\frac{\partial P'}{\partial r} \quad \text{for} \quad 0 \leqslant r < a \tag{20}$$

and

$$N = \frac{1}{\rho_0\omega_0^2(1 + \alpha M_0/k_0)^2}\frac{\partial P'}{\partial r} \quad \text{for} \quad r > a. \tag{21}$$

The solution for P' can then be written down and p' deduced (in the far field) by the method of stationary phase. We give the expression for p' as

$$p' \sim \frac{-i\rho_0 Q_{xx}^0\,\omega_0^2 \exp[i(\omega_0 t - k_0 R)]\cos^2\theta}{2\pi^2 R(1 - M_0\cos\theta)^3 c_0^2\{H_0^{(2)}(\alpha^+ a)\,I_0'(\hat{\alpha}^+ a)\,(\hat{\alpha}^+ a)\,(\rho_0/\rho_1) - (\alpha^+ a)\,H_0^{(2)\prime}(\alpha^+ a)\,I_0(\hat{\alpha}^+ a)\,(1 - M_0\cos\theta)^2\}} \tag{22}$$

for $0 \leqslant \theta \leqslant \cos^{-1}[(c_1/c_0 + M_0)]^{-1}$, where $\alpha^+ = k_0\sin\theta/(1 - M_0\cos\theta)$ and $\hat{\alpha}^+$ is the positive square root $(k_1^2 - \alpha^2)^{\frac{1}{2}}$ with $\alpha = k_0\cos\theta/(1 - M_0\cos\theta)$. For

$$\cos^{-1}[(c_1/c_0 + M_0)^{-1}] \leqslant \theta \leqslant \pi,$$

the same expression applies with $\hat{\alpha}^+$ replaced by $\tilde{\alpha}^+$, which is the positive square root $(\alpha^2 - k_1^2)^{\frac{1}{2}}$, and the I's replaced by J's. The procedures for deriving the solutions for x-y, y-z and y-y quadrupoles carry over more or less mechanically from part 1 with the only differences that $\hat{\alpha}^+$ or α^+ are now $|k_1^2 - \alpha^2|^{\frac{1}{2}}$, the zone of silence now ranges over $0 \leqslant \theta \leqslant \cos^{-1}[(c_1/c_0 + M_0)^{-1}]$ and a factor ρ_0/ρ_1 multiplies the term involving I' or J' in the denominator of expressions such as (22).

The additional solutions that need to be worked out correspond to (*a*) an axial-dipole solution, i.e. with a source term of the form

$$\partial[\delta(x - Vt)\,\delta(y)\,\delta(z)]/\partial x,$$

(*b*) a radial dipole term of the form

$$\partial[\delta(x - Vt)\,\delta(y)\,\delta(z)]/\partial r$$

and (*c*) a pressure source term of the form

$$\delta(x - Vt)\,\delta(y)\,\delta(z).$$

These solutions are just as easy to derive as the quadrupole solutions and hence only the broad outlines will be sketched. The solution for (*a*) is very similar to (22) with the factor $\omega_0^2/[c_0^2(1 - M\cos\theta)^2]$ (or α^2) replaced by just $\omega_0 \cos\theta/[c_0(1 - M\cos\theta)]$ (or α). The solution for (*b*) is similar to that for (*a*) with α replaced by $\tilde{\alpha}^+$ or $\hat{\alpha}^+$ in the numerator and the Bessel functions of order zero replaced by those of order one. Finally the solution for (*c*) is identical to that for (*a*) except that no term α appears in the numerator.

3. Physical interpretation of the solutions and applications to jet-noise experiments

At least three distinct mechanisms affecting jet noise can be identified as influencing the radiation by quadrupoles in a hot jet. First, for the transverse quadrupoles (as in part 1) the mechanisms of phase cancellation or the Stokes effect are now governed by the flow properties within the jet. Since the speed of sound is higher within the jet than outside it, the Stokes effect tends to diminish the radiative efficiency of the transverse quadrupoles as the jet temperature is increased. Second, especially at low frequencies there is a transmission or dynamic density effect tending to enhance the radiation by a factor ρ_0/ρ_1. Ribner (1964) alludes to this by considering the problem of a monopole source of strength $Q_0 \exp(i\omega_0 t)$ embedded in a sphere of gas of density ρ_1 and sound speed c_1 (and of radius a) with the ambient fluid at a density ρ_0 and speed of sound, c_0. In the limit of $\omega_0 a/c_0 \to 0$, the source appears to the ambient fluid as if it were of strength $Q_0\rho_0/\rho_1$. Q_0 in this instance is a source strength with dimensions mass/time. If the cause of Q_0, for instance, was a pulsating sphere undergoing a fixed volumetric displacement, Q_0 itself would be proportional to ρ_1 and one might then restate this result as tantamount to the observation that a pulsating sphere of fixed volumetric displacement embedded in a gas different from the ambient fluid radiates a field independent of the local density provided that the region of inhomogeneous density is compact. Equation (22) manifests this same result because in the limit $k_0 a \to 0$, p' is proportional to ρ_0 even though the strength of the x-x quadrupole was taken as proportional to ρ_1 [see (14)]. Finally, Lilley's equation shows that, while the strength of the quadrupoles themselves varies as ρ_1 (and hence diminishes as the jet temperature increases), there are associated with the transverse singularities additional dipole and source-like mechanisms related to the density gradients of the mean flow which increase with increasing jet temperature.

It is worthy of note that the Lighthill expression (equation (3) of part 1) is not now identifiable as any valid limit whether at low Mach numbers or low frequencies or even at $\theta = 90°$. At the 90° point, jet-flow shrouding effects are present for hot jets simply because a temperature inhomogeneity is a scalar inhomogeneity, unlike a velocity inhomogeneity, so that there is no question of there not being a component at $\theta = 90°$. Besides, the 90° radiation is dominated by the transverse singularities which generate the additional source-like and dipole-like terms not accounted for by Lighthill's expression. Indeed, of the various agencies identified as governing the radiation by quadrupoles in a hot jet, the Lighthill expression picks up only one effect, namely the variation of the quadrupole strength as ρ_j. Perhaps it can now be appreciated why it was mentioned earlier that the area of hot-jet noise is one in which equations such as Lilley's yield insight far removed from that provided by Lighthill's theory.

We now turn to the problem of the application of the above to jet-noise data. The x-y and x-z quadrupoles each generate an additional dipole while the y-y, y-z and z-z quadrupoles generate two additional terms. This gives rise to fourteen primary solutions (unlike the six dealt with by Ribner 1969). On constructing an expression for the mean-square pressure, 196 types of interaction terms arise though several of these will undoubtedly vanish upon averaging circumferentially. To avoid too complicated a solution procedure, the formula employed for cold jets [equation (34) of part 1] was used in the current study too with one additional simplification. Since representative average estimates were made of $\overline{d\rho/dr}$, $\overline{d^2\rho/dr^2}$, etc., whenever a quadrupole singularity generates additional dipole-like and source-like terms, interference between multipole singularities of different order was also neglected. In other words, multipole singularities of each order were assumed to contribute independently to the mean-square pressure due to that quadrupole. Circumferential averaging in the ϕ direction was also carried out as usual.

In accordance with the assumption about the velocity of the plug-flow jet used to compare the predictions of the theory with experiment in part 1, the temperature T_1 of the plug-flow jet was taken as

$$0{\cdot}65 \times (\text{ideal-jet exit temperature}) + 0{\cdot}35 \times (\text{ambient temperature}).$$

The density ρ_1 was calculated correspondingly from $\rho_1 T_1 = \rho_0 T_0$.

One last assumption was made in connexion with the estimation of density gradients and needs some discussion. The relative contributions of the quadrupoles, dipoles and simple sources appear (for given velocity fluctuation levels) to scale as

$$\bar{\rho} k_0^2, \quad (d\bar{\rho}/dr)\, k_0, \quad d^2\bar{\rho}/dr^2. \tag{23}$$

This suggests that density gradients primarily influence the low frequency sound at a given jet velocity. It might even appear to explain the experimentally observed progressive bias towards the lower frequencies of the pressure spectra when the temperature of a constant velocity jet is raised. We believe that such an inference is erroneous for the following reason. It must be recognized that the low frequency sound is likely to be emitted from a region where the density gradients are small and conversely the high frequency sound from the regions of the flow where the gradients are large. This notion can be empirically stated as

$$\frac{d\bar{\rho}}{dr} = \frac{(\rho_0-\rho_1)}{a}(\text{source Strouhal number})\, C_1 \tag{24}$$

and

$$\frac{d^2\bar{\rho}}{dr^2} = \frac{(\rho_0-\rho_1)}{a^2}(\text{source Strouhal number})^2 C_2 \tag{25}$$

(C_1 and C_2 are non-dimensional constants to be specified shortly). Equations (24) and (25) attempt to recognize that the low Strouhal number emission occurs further downstream in a jet and conversely high Strouhal number emission occurs close to the nozzle exit. With the aid of (24) and (25) and reinstating the velocity-fluctuation dependence in (23), for a given source Strouhal number the quadrupole, dipole and source terms should scale as

$$(\text{jet velocity})^4\,\{\rho_1, (\rho_0-\rho_1)/M_0, (\rho_0-\rho_1)/M_0^2\}. \tag{26}$$

Equation (26) reveals [as does (23)] that there are 'excess' noise mechanisms formally scaling as M_0^6 and M_0^4 (for the intensity) when density gradients are allowed for. This is, of course, merely a reflexion of the dipole and source-like nature of the additional singularities introduced by the density gradients. The situation with regard to a power law for the velocity is of course not very clear-cut since often density gradients and velocities are changed simultaneously in experiments. Equation (26) does show however that, if the velocity of a hot jet is changed at constant jet temperature, there will be M_0^6- and M_0^4-type noise contributions owing to the coupling of the velocity fluctuations and density gradients. Interestingly enough, Bushell (1971) has found in an effort to correlate high bypass ratio fan engine noise that the noise of the colder outer fan jet follows an eighth-power law down to much lower velocities than the noise of the hot core jet. Equation (26) also confirms that at constant M_0 changing jet temperature does not affect the mixture of the source terms proportional to ρ_j and $\rho_0-\rho_j$ differently for different Strouhal numbers. Thus it cannot explain the bias of the power spectra progressively towards lower frequencies owing to heating. This appears to be a propagation effect. With regard to (24) and (25), one data point for $M_0 = 0{\cdot}4$ at $\theta = 90°$ from Tanna (1973) was used to establish that $C_1 = C_2 = 1$ appears to suffice to explain those data. These values have subsequently been used in all the comparisons with the data of Tanna (1973), Hoch *et al.* (1972) and Hoch (1974, private communication).

Two comparisons have been carried out with available data. As in part 1, the theory is restricted to jet velocities such that only subsonic eddy convection velocities are involved (i.e. jet velocities less than $1/0{\cdot}65$ times the atmospheric speed of sound). In figure 4, we compute the variation of the total power of the

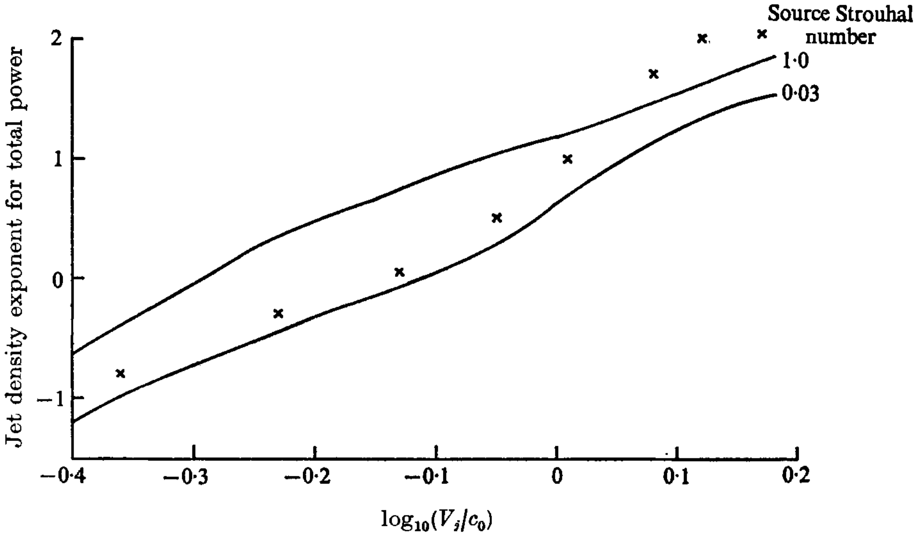

FIGURE 4. Jet density exponent for total power as a function of V_j/c_0. ×, OAPWL data from Hoch *et al.*; ——, present theory for indicated source Strouhal numbers.

directivity pattern at constant jet velocity for various source Strouhal numbers (from 0·03 to 1·0) as a function of ρ_j/ρ_0 and exhibit the result as the jet density exponent ω, i.e. the acoustic power varies as ρ_j^ω for constant jet velocity. Hoch *et al.* (1972) have given such data for the overall power level and, as figure 4 shows, the current theory brackets the data quite well when one considers the theoretical predictions for source Strouhal numbers of 0·03 and 1·0 (which should bracket the dominant eddy frequency range quite well). Especially at lower jet velocities, the theory also predicts a progressive bias of the power spectrum towards lower frequencies due to heating (the index at a source Strouhal number of 0·03 is less than that at 1·0), which is also in accord with the study of Hoch *et al.* (1972). The available data on hot-jet noise are rather sparse and not yet available in the form of directivity plots at constant source Strouhal numbers, which is the sort of information that one can directly compare against the predictions of an acoustic theory.

Figures 5–10 show computed exponents at various angles from the jet axis for various jet velocities. Since we are dealing with a fixed angle now, the data can be exhibited in terms of observed Strouhal numbers: a more recognizable quantity than a source Strouhal number. The exponents have been computed for observed Strouhal numbers ranging from 0·1 to 1·0. Such calculations were always carried out by fitting a straight line by the least-squares method to a plot of $\log\langle p^2\rangle$ against $\log(\rho_j/\rho_0)$. The calculations were performed for ρ_j/ρ_0 ranging from 0·2 to 1·0. Shown by crosses are the data for the exponent for the OASPL as measured by Hoch (1974, private communication). The broad agreement is again most encouraging. Note that at shallow angles to the jet axis the dominant Strouhal numbers observed for the pressure spectrum of a cold jet lie between 0·1 and 0·3 with values between 0·6 and 1·0 being more typical of the spectra at the larger angles ('reverse' Doppler shift). Thus it is quite appropriate that the experimental

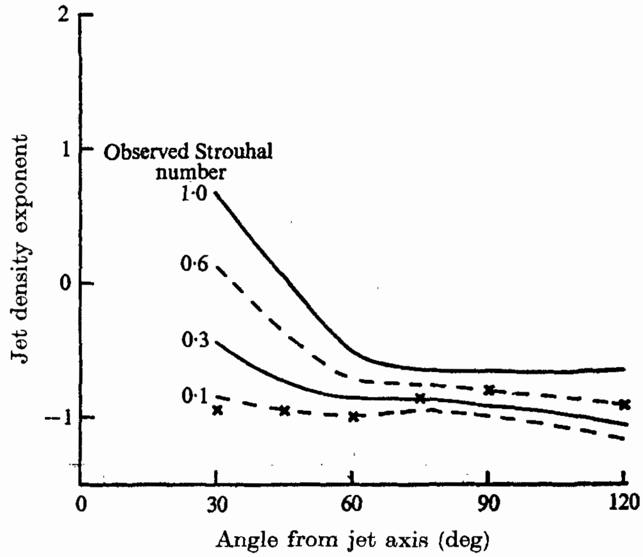

FIGURE 5. Jet density exponent for $V_j/c_0 = 0{\cdot}447$ as a function of the angle from the jet axis. ×, OASPL data from Hoch *et al.*

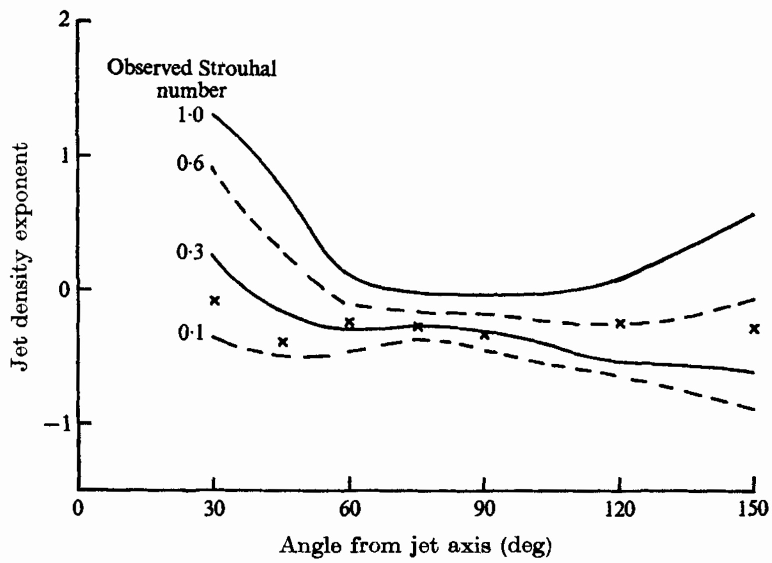

FIGURE 6. Jet density exponent for $V_j/c_0 = 0{\cdot}589$ as a function of the angle from the jet axis. ×, OASPL data from Hoch *et al.*

data for the OASPL should agree better with the lower frequency curves at shallow angles and with the higher frequency curves at larger angles. The high, positive exponents at shallow angles for high frequencies are a manifestation of the rapid diminution of sound pressure at these angles and frequencies owing to heating because of the refractive sweepback of the pattern upon heating the jet. (Recall the enlargement of the zone of 'silence' from $0 \leqslant \theta \leqslant \cos^{-1}[(1+M)^{-1}]$ to $0 \leqslant \theta \leqslant \cos^{-1}\{[(c_1/c_0)+M_0)]^{-1}\}$ with heating.) Figure 11 is essentially a replot

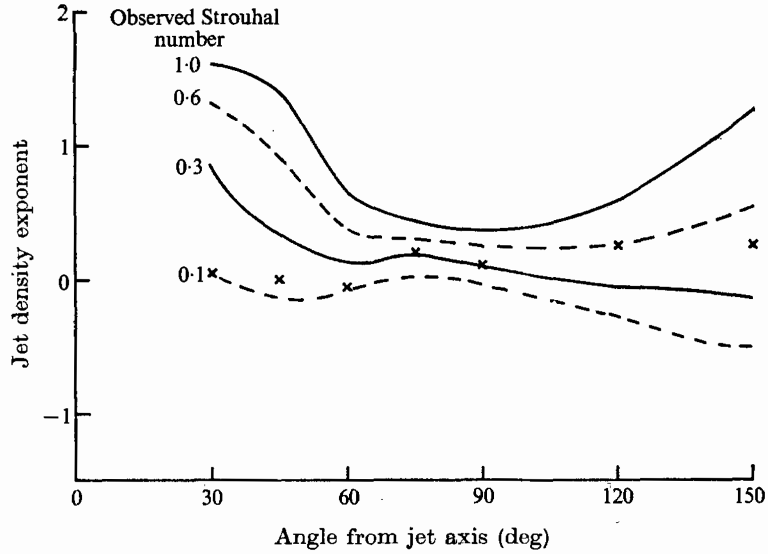

FIGURE 7. Jet density exponent for $V_j/c_0 = 0{\cdot}741$ as a function of the angle from the jet axis. ×, OASPL data from Hoch *et al.*

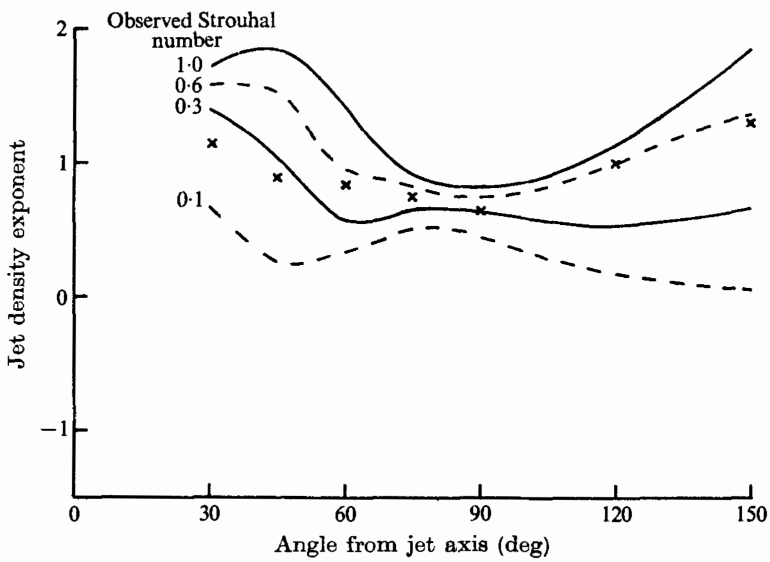

FIGURE 8. Jet density exponent for $V_j/c_0 = 0{\cdot}891$ as a function of the angle from the jet axis. ×, OASPL data fron Hoch *et al.*

of the 90° data in figures 5–10 since experimentalists often concentrate on this measurement station. With hot jets, however, we have shown that the 90° location is by no means a clear indicator of the turbulence source strength and spectrum variation and thus there seems little that is unique about such a location (except that convection effects are always absent here). Figure 11 indicates an apparent overprediction of the exponent by the theory at the high velocity end. In figure 12, however, we show comparisons with exponents computed from

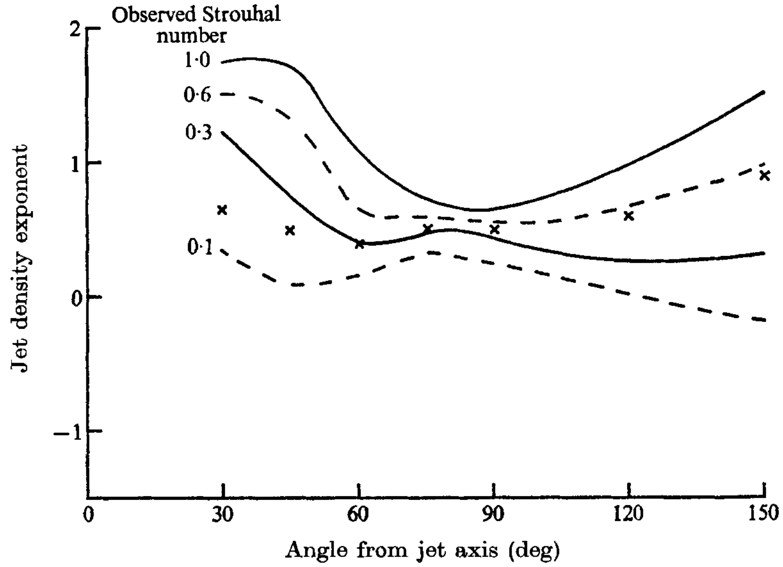

FIGURE 9. Jet density exponent for $V_j/c_0 = 1{\cdot}023$ as a function of the angle from the jet axis. ×, OASPL data from Hoch *et al.*

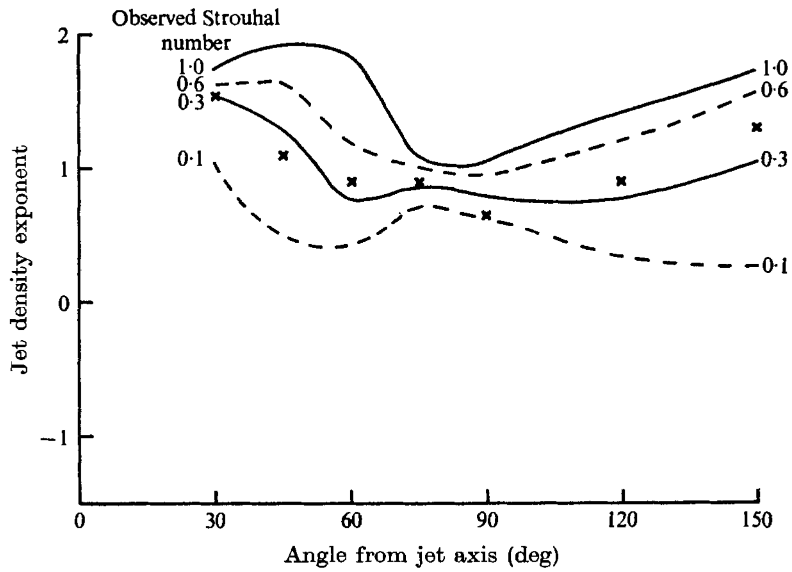

FIGURE 10. Jet density exponent for $V_j/c_0 = 1{\cdot}175$ as a function of the angle from the jet axis. ×, OASPL data from Hoch *et al.*

the data of Tanna (1973) for three different jet velocities and for several frequencies. The data–theory comparison at $M_0 = 0{\cdot}4$ should be discounted as the choice of C_1 and C_2 in (24) was based on these data. The theory predicts the data at $M_0 = 0{\cdot}8$ quite well but now underpredicts the data at $M_0 = 1{\cdot}47$. Thus, considering the three sets of data of Hoch *et al.* (1972), Hoch (1974) and Tanna (1973), it is difficult to detect systematic failure of the theory.

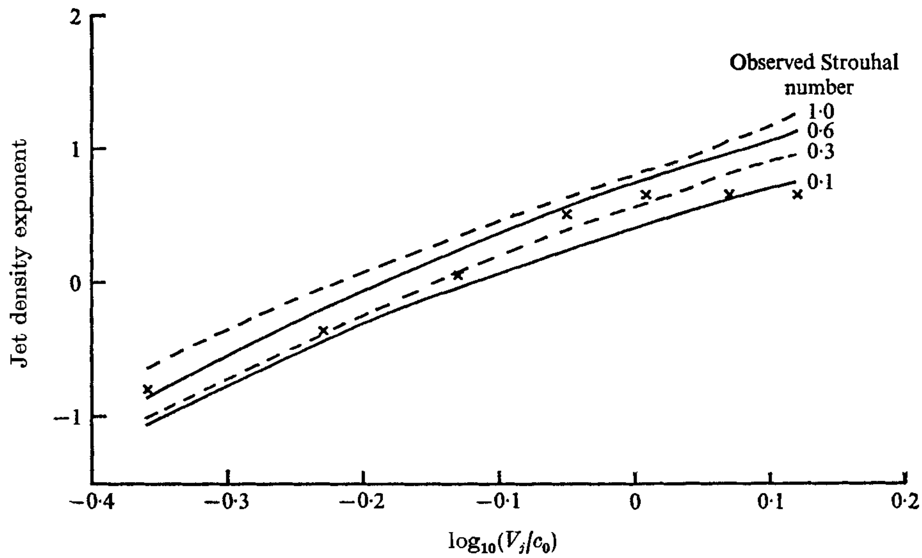

FIGURE 11. Jet density at 90° to jet axis as a function of V_j/c_0. ×, OASPL data from Hoch *et al.*

As with the results of part 1, this ability to predict the data suggests both the solution of the problem of scaling the effects of jet temperature on jet noise and the conclusion that there is very little temperature effect on the turbulence source spectrum so long as the jet velocity is held fixed.

4. Concluding remarks

The problem of noise generation by a heated jet has been systematically studied with the aid of Lilley's equation. Only the velocity fluctuations have been considered as the source of the sound but mean density gradients act to generate dipole and simple-source terms which produce jet noise scaling with jet velocity as M^6 and M^4 for constant values of (jet temperature − ambient temperature). Such additional singular source-like terms arise only for the transverse quadrupoles.

The problem of heated-jet noise, especially at low jet velocities, is one in which Lighthill's analysis of jet noise offers very little guidance. The physical picture of jet noise due to Lighthill, of compact eddies convecting and decaying with the flow, still carries over of course. The enhancement of the Stokes effect (leading to inefficiency of quadrupole radiation) for the transverse quadrupoles, the transmission of low frequency sound across density gradients and the generation of new dipole and source-like terms are all features governing the noise of heated jets. The progressive bias of the acoustic pressure and power spectra towards lower frequencies owing to heating at constant jet exit velocity appears to be primarily a propagation effect. In no sense can one identify the Lighthill expression for jet noise as a valid limit in the area of heated-jet noise.

Extensive comparisons with data have been carried out and a very wide measure of agreement achieved for the detailed effects on the total power and on the sound

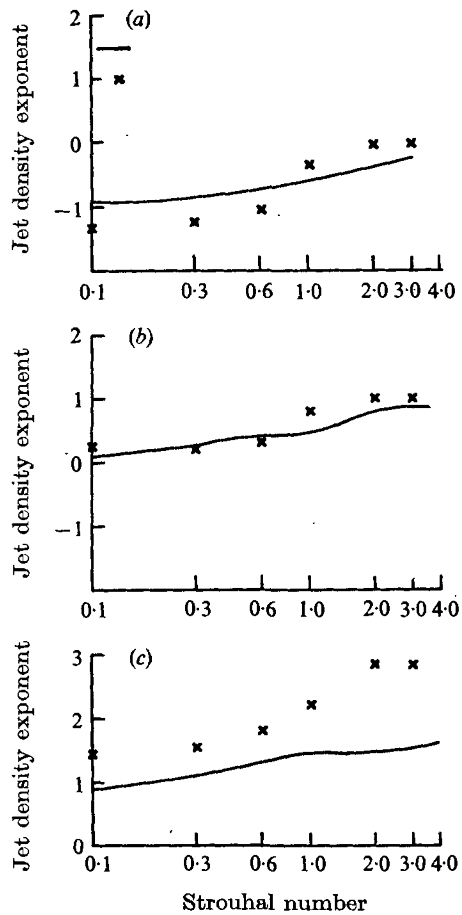

FIGURE 12. Jet density exponent at 90° to jet axis for (a) $V_j/c_0 = 0{\cdot}4$, (b) $V_j/c_0 = 0{\cdot}8$ and (c) $V_j/c_0 = 1{\cdot}47$ as a function of frequency. ×, Lockheed data from Tanna (1973).

pressure at the various angles over a wide range of jet velocities. Apart from some explicitly stated assumptions, the theory assumes no change in the intrinsic non-dimensional turbulence source spectrum owing to variations in jet temperature at constant jet exit velocity. To the extent to which such a theory is able to explain the data then, we must question whether any significant jet temperature effects exist in so far as the quadrupole source distributions are concerned. The results of part 1 have shown a comparable absence of jet Mach number effects on this distribution.

Taken together, parts 1 and 2 of the current study have essentially solved the problem of scaling jet noise on changes in jet velocity and jet temperature at various frequencies and angles to the jet axis. Nozzle size effects, of course, have always been a well-understood phenomenon. It is worthy of note that the idea of convected compact quadrupoles being the primary noise generators has proved quite adequate in this whole study of jet noise due to jet velocities up to about 1·5 times the atmospheric speed of sound.

I am most deeply indebted to Dr Brian Tester of the Lockheed Corporation at Georgia for his drawing my attention to the role of mean density gradients in the generation of noise from heated jets. Very special thanks are also due to Mr R. G. Hoch of SNECMA for his supplying me with unpublished data on jet density exponents for the OASPL displayed in figures 5–11. I wish to acknowledge most gratefully the benefit derived from innumerable discussions, correspondence, etc. with Professor John E. Ffowcs Williams, Professor H. S. Ribner and my colleague Dr Thomas F. Balsa. The study was supported financially by the U.S. Department of Transportation and the U.S. Air Force under contract F33615-73-C-2031.

REFERENCES

Bushell, K. W. 1971 *J. Sound Vib.* **17**, 271.

Hoch, R. G., Duponchel, J. P., Cocking, B. J. & Bryce, W. D. 1972 *J. Sound Vib.* **28**, 649.

Lighthill, M. J. 1959 *Introduction to Fourier Analysis and Generalised Functions.* Cambridge University Press.

Morfey, C. L. 1973 *J. Sound Vib.* **31**, 391.

Ribner, H. S. 1964 *Adv. Appl. Mech.* **8**, 103–182.

Ribner, H. S. 1969 *J. Fluid Mech.* **38**, 1.

Tanna, H. K. 1973 *A.I.A.A. Paper*, no. 73–991.